Available online at www.sciencedirect.com

ScienceDirect

Journal of the Franklin Institute 355 (2018) 4839–4853

www.elsevier.com/locate/jfranklin

Optimal control of stochastic functional neutral differential equations with time lag in control☆

Kai Liu[a,b,*]

[a] College of Mathematical Sciences, Tianjin Normal University, Tianjin 300387, PR China

[b] Department of Mathematical Sciences, School of Physical Sciences, The University of Liverpool, Liverpool L69 7ZL, UK

Received 27 November 2017; received in revised form 23 February 2018; accepted 22 April 2018

Available online 28 May 2018

Abstract

In this work, we consider an optimal control problem of a class of stochastic differential equations driven by additive noise with aftereffect appearing in control. We develop a semigroup theory of the driving deterministic neutral system and identify explicitly the adjoint operator of the corresponding infinitesimal generator. We formulate the time delay equation under consideration into an infinite dimensional stochastic control system without time lag by means of the adjoint theory established. Consequently, we can deal with the associated optimal control problem through the study of a Hamilton–Jacob–Bellman (HJB) equation. Last, we present an example whose optimal control can be explicitly determined to illustrate our theory.

1. Introduction

Many physical phenomena can be modeled by stochastic dynamical systems whose evolution in time is governed by random forces as well as intrinsic dependence of the

☆ The author is grateful to the Tianjin Thousand Talents Plan for its financial support.

* Correspondence to: Department of Mathematical Sciences, School of Physical Sciences, The University of Liverpool, Liverpool L69 7ZL, UK.

E-mail address: k.liu@liverpool.ac.uk.

https://doi.org/10.1016/j.jfranklin.2018.04.044

state on a finite period of its past history. In those dynamical systems with aftereffect, neutral-type systems, deterministic or stochastic, play a special role in both theory and practical application. In the meanwhile, there exist extensive researches devoted to those stochastic control systems with time delay in the controller's design, e.g., see Hale and Lunel [6], Kolmanovskii and Myshkis [10], Mao [12], Salamon [13] and references therein.

To present a motivational example, let us mention the transmission line equation (see [1])

$$\begin{cases} a\dfrac{\partial y(t,\xi)}{\partial t} + \dfrac{\partial z(t,\xi)}{\partial \xi} = 0, \\ \dfrac{1}{a}\dfrac{\partial z(t,\xi)}{\partial t} + \dfrac{\partial y(t,\xi)}{\partial \xi} = 0, \quad t \geq 0, \quad \xi \in [0,1], \end{cases} \tag{1.1}$$

with initial condition $y(0,\xi) = y_0(\xi)$, $z(0,\xi) = z_0(\xi)$ where $a > 0$ is a constant. Let

$$\begin{pmatrix} w_1(t,\xi) \\ w_2(t,\xi) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{a} & \sqrt{a} \\ -1/\sqrt{a} & \sqrt{a} \end{pmatrix} \begin{pmatrix} z(t,\xi) \\ y(t,\xi) \end{pmatrix}, \tag{1.2}$$

then (w_1, w_2) satisfies the equation

$$\frac{\partial}{\partial t}\begin{pmatrix} w_1(t,\xi) \\ w_2(t,\xi) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial w_1(t,\xi)/\partial\xi \\ \partial w_2(t,\xi)/\partial\xi \end{pmatrix}. \tag{1.3}$$

It may be shown that there exists a unique solution (w_1, w_2) to Eq. (1.3). In practice, there are boundary noise conditions which are determined by lump terminal networks. One possible set of these kinds of conditions is like

$$dy(t,0) + r_1 y(t,0)dt + r_2 dw(t) = z(t,0)dt, \; z(t,1) = 0, \tag{1.4}$$

where $r_i \in \mathbb{R}$, $i = 1, 2$, and $w(t)$, $t \geq 0$, is a real standard Brownian motion on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Upon consideration of a control problem of Eq. (1.1), it has been noticed that there are occasions when a controller with time delay may quench the oscillation, yielding a smooth and fast transient response. This consideration basically leads to a controlled system with a retarded boundary control of the form

$$dy(t,0) + r_1 y(t,0)dt + bv(t-r)dt + r_2 dw(t) = z(t,0)dt, \; z(t,1) = 0, \tag{1.5}$$

where $b \in \mathbb{R}$, $r > 0$ and $v(t)$, $t \geq -r$, is the control for the overall system. Let us define $x_1(t) = w_1(t,0)$ and $x_2(t) = w_2(t,1)$, $t \geq 0$. Upon substitution of these new terms into Eq. (1.5), one can derive the following stochastic delay differential equation of neutral type

$$\begin{cases} d(x_1(t) + x_2(t-r)) + \alpha(x_1(t) + x_2(t-r))dt = \beta(x_1(t) - x_2(t-r))dt + \gamma v(t-r)dt \\ \quad + \delta dw(t), \quad t \geq 0, \\ x_1(t-r) - x_2(t) = 0, \qquad t \geq 0, \end{cases} \tag{1.6}$$

where $\alpha, \beta, \gamma, \delta$ are constants which can be determined by r_1, r_2 and a. Basically, we have from Eq. (1.6) that $x_2(t) = x_1(t-r)$, $t \geq 0$, and let $u(t) = v(t+r)$, $t \geq -2r$. Substitution of these into Eq. (1.6) yields further a stochastic neutral differential equation with respect to $x(t) \equiv x_1(t)$, $t \geq -2r$,

$$d(x(t) + x(t-2r)) = a_0 x(t)dt + a_1 x(t-2r)dt + b_0 u(t-2r)dt + c_0 dw(t), \tag{1.7}$$

where a_0, a_1, b_0, $c_0 \in \mathbb{R}$, with initial condition

$$\begin{cases} x(t) = \dfrac{z_0(-t)}{\sqrt{a}} + \sqrt{a}y_0(-t), & -r \le t \le 0, \\ x(t) = -\dfrac{z_0(2r+t)}{\sqrt{a}} + \sqrt{a}y_0(2r+t), & -2r \le t < -r. \end{cases} \tag{1.8}$$

Note that for $0 \le \xi \le 1$ and $t \ge 0$,

$$\begin{cases} z(t,\xi) = \dfrac{\sqrt{a}}{2}(x(t-\xi) - x(t+\xi-2r)), \\ y(t,\xi) = \dfrac{1}{2\sqrt{a}}(x(t-\xi) + x(t+\xi-2r)). \end{cases} \tag{1.9}$$

For the given system (1.1) and (1.5), let $H = L^2(0,1)$ and one would like to minimize the cost functional

$$J = \mathbb{E}^{\mathbb{P}}\left\{ q_1\|y(T,\cdot)\|_H^2 + q_2\int_0^T u^2(t)dt \right\}, \qquad T \ge 0,$$

where $q_i > 0$, $i = 1, 2$. Upon the transformation Eq. (1.9), it can be equally formulated as a control problem for Eq. (1.7) with the cost functional

$$J = \mathbb{E}^{\mathbb{P}}\left\{ \frac{q_1}{4a}\int_0^1 \left[x(T-\xi) + x(T+\xi-2r)\right]^2 d\xi + q_2\int_0^T u^2(t)dt \right\}, \qquad T \ge 0.$$

The problem we are concerned about here is the minimization of the objective functional $J(\cdot)$ over all the so-called admissible controls u.

The remarkable feature in the above stochastic control system is for the time delay to appear in the control term, a fact which generally causes serious difficulty in the study of control problems. We shall handle this trouble, however, by using some techniques from stochastic optimal control in infinite dimensions. In Vinter and Kwong [14], they handled this phenomenon for non-neutral, deterministic systems, and Gozzi et al. [5] generalized their theory to a class of non-neutral stochastic systems by associating a controlled stochastic differential equation with delay both in the state and control to a stochastic control problem without time delay in a suitable infinite dimensional space. In the same spirit, we shall extend in this paper their work to consider the optimal control problem of stochastic functional differential equations of neutral type.

This work is organized as follows. In Section 2, we first present a semigroup theory which allows us to lift the optimal control problem of a neutral type time delay system to an infinite dimensional control problem without time delay. In this section, we also develop an adjoint theory of the driving deterministic neutral system under consideration. This formulation will allow us in Section 3 to establish a stochastic control theory of a system with delay in both the state and control terms by developing some verification results. Finally, in Section 4 we consider an example of a controlled SDDE of neutral type whose corresponding HJB equation admits an explicit solution; hence there exists an optimal control in feedback form for the control problem.

2. Generator and its adjoint theory

Let $\mathbb{R}^n$ be the n-dimensional real vector space equipped with the usual Euclidean norm $\|\cdot\|$ and point product $\langle\cdot,\cdot\rangle$, respectively. Let $\mathcal{M}(\mathbb{R}^n,\mathbb{R}^m)$ denote the family of all $n \times m$ real matrices. If $m = n$, we simply write $\mathcal{M}(\mathbb{R}^n)$ for $\mathcal{M}(\mathbb{R}^n, \mathbb{R}^n)$. Let $r > 0$ and $\mathcal{H} = \mathbb{R}^n \times L^2([-r, 0], \mathbb{R}^n)$. For any $T \geq 0$ and $x \in L^2([-r, T], \mathbb{R}^n)$, we always write $x_t(\theta) := x(t + \theta)$ for any $t \geq 0$ and $\theta \in [-r, 0]$.

We introduce two linear mappings D and F on $C([-r, 0], \mathbb{R}^n)$, respectively, by

$$D\varphi := \int_{-r}^{0} d\eta(\theta)\varphi(\theta), \qquad F\varphi := \int_{-r}^{0} d\bar{\eta}(\theta)\varphi(\theta), \qquad \forall \varphi(\cdot) \in C([-r, 0], \mathbb{R}^n),$$

where η, $\bar{\eta}$ are matrix-valued functions of bounded variation which vanish at $\theta = 0$ and are left-continuous in $(-r, 0)$. It is shown (see, e.g., [11]) that mappings D and F have a linear extension to $L^2([-r, T], \mathbb{R}^n)$ for any $T \geq 0$, still denote them by D and F, such that

$$\int_0^T \|Dx_t\|^2 dt \leq C_1 \int_{-r}^{T} \|x(t)\|^2 dt, \qquad \forall x \in L^2([-r, T], \mathbb{R}^n), \tag{2.1}$$

and

$$\int_0^T \|Fx_t\|^2 dt \leq C_2 \int_{-r}^{T} \|x(t)\|^2 dt, \qquad \forall x \in L^2([-r, T], \mathbb{R}^n), \tag{2.2}$$

where $C_1 > 0$, $C_2 > 0$ are two constants.

Given arbitrarily $\phi = (\phi_0, \phi_1) \in \mathcal{H}$ and $T \geq 0$, let us consider a deterministic functional differential equation of neutral type in $\mathbb{R}^n$,

$$\begin{cases} x(t) = \phi_0 + Dx_t + \displaystyle\int_0^t Fx_s ds, \qquad t \in [0, T], \\ x_0 = \phi_1. \end{cases} \tag{2.3}$$

It is known (cf. [7]) that there exists a unique solution $x \in L^2([-r, T], \mathbb{R}^n)$ for any $T \geq 0$ which satisfies $x_0 = \phi_1$ in L^2 and the first equation in Eq. (2.3) holds almost everywhere.

In association with the unique solution $x(t, \phi)$, $t \geq -r$, of system (2.3), we define a family of bounded linear operators $\mathcal{S}(t) : \mathcal{H} \to \mathcal{H}$, $t \geq 0$, by

$$\mathcal{S}(t)\phi = (x(t) - Dx_t, x_t) \quad \text{for any} \quad \phi \in \mathcal{H}. \tag{2.4}$$

It was shown (see, e.g., [2,7–9]) that $\mathcal{S}(t)$, $t \geq 0$, is a C_0-semigroup on $\mathcal{H}$ and its generator $\mathcal{A}$ is described by

$$\mathscr{D}(\mathcal{A}) = \Big\{(\phi_0, \phi_1) \in \mathcal{H} : \phi_1 \in W^{1,2}([-r, 0], \mathbb{R}^n),\ \phi_0 = \phi_1(0) - D\phi_1\Big\}$$

and for each $\phi = (\phi_0, \phi_1) \in \mathscr{D}(\mathcal{A})$, $\mathcal{A}\phi = (F\phi_1, \phi_1') \in \mathcal{H}$. Here $W^{1,2}([-r, 0], \mathbb{R}^n)$ is the standard Sobolev space.

For each $\lambda \in \mathbb{C}$, we define an $n \times n$ matrix $D(e^{\lambda \cdot})$ by $D(e^{\lambda \cdot})x = D(e^{\lambda \cdot}x)$ for $x \in \mathbb{R}^n$, and matrix $F(e^{\lambda \cdot})$ is similarly defined. Further, for each $\lambda \in \mathbb{C}$, we define an $n \times n$ matrix $\Delta(\lambda)$ by

$$\Delta(\lambda) = \lambda(I - D(e^{\lambda \cdot})) - F(e^{\lambda \cdot})$$

where I is the $n \times n$ identity matrix. The resolvent set $\rho(D, F)$ is defined as the set of all values $\lambda \in \mathbb{C}$ for which the matrix $\Delta(\lambda)$ has an invertible matrix $\Delta(\lambda)^{-1}$ on $\mathbb{R}^n$.

The following proposition whose proof is referred to Ito and Tarn [7] or Liu [11] states relationship between resolvent $\Delta(\lambda)^{-1}$ and the resolvent operator of $\mathcal{A}$.

Proposition 2.1. *Let $\lambda \in \mathbb{R}$ and $\varphi = (\varphi_0, \varphi_1) \in \mathcal{H}$. If $\phi = (\phi_0, \phi_1) \in \mathscr{D}(\mathcal{A})$ satisfies*

$$\lambda\phi - \mathcal{A}\phi = \varphi, \tag{2.5}$$

then

$$\phi_1(\theta) = e^{\lambda\theta}\phi_1(0) + \int_\theta^0 e^{\lambda(\theta-\tau)}\varphi_1(\tau)d\tau, \qquad -r \le \theta \le 0, \tag{2.6}$$

and

$$\Delta(\lambda)\phi_1(0) = \lambda D\bigg(\int_\cdot^0 e^{\lambda(\cdot-\tau)}\varphi_1(\tau)d\tau\bigg) + F\bigg(\int_\cdot^0 e^{\lambda(\cdot-\tau)}\varphi_1(\tau)d\tau\bigg) + \varphi_0. \tag{2.7}$$

Conversely, if $\phi_1(0) \in \mathbb{R}^n$ satisfies Eq. (2.7) and let $\phi_0 = \phi_1(0) - D\phi_1$ where

$$\phi_1(\theta) = e^{\lambda\theta}\phi_1(0) + \int_\theta^0 e^{\lambda(\theta-\tau)}\varphi_1(\tau)d\tau, \qquad -r \le \theta \le 0, \tag{2.8}$$

then we have that $\phi_1 \in W^{1,2}([-r, 0], \mathbb{R}^n)$, $\phi = (\phi_0, \phi_1) \in \mathscr{D}(\mathcal{A})$ and ϕ satisfies Eq. (2.5).

Now we are interested in the "adjoint" system of Eq. (2.3). Precisely, let η^T and $\bar{\eta}^T$ denote the transpose matrix of η and $\bar{\eta}$, respectively. We introduce two linear mappings D^T and F^T on $C([-r, 0], \mathbb{R}^n)$ by

$$D^T\varphi = \int_{-r}^0 d\eta^T(\theta)\varphi(\theta), \qquad F^T\varphi = \int_{-r}^0 d\bar{\eta}^T(\theta)\varphi(\theta), \qquad \forall\, \varphi \in C([-r, 0], \mathbb{R}^n).$$

In a similar manner, it can be shown that both D^T and F^T have a bounded linear extension on $L^2([-r, T], \mathbb{R}^n)$ for any $T \ge 0$. Given arbitrarily $\zeta = (\zeta_0, \zeta_1) \in \mathcal{H}$, we can similarly consider the following deterministic differential equations of neutral type in $\mathbb{R}^n$,

$$\begin{cases} x(t) = \zeta_0 + D^T x_t + \int_0^t F^T x_s ds, & t \in [0, T], \\ x_0 = \zeta_1. \end{cases} \tag{2.9}$$

It is also shown that $\mathcal{A}^*$, the adjoint of $\mathcal{A}$, generates a C_0-semigroup $e^{t\mathcal{A}^*}$, $t \ge 0$, on $\mathcal{H}$. For $\lambda \in \mathbb{C}$, we define an $n \times n$ matrix $\Delta_T(\lambda)$ by

$$\Delta_T(\lambda) = \lambda(I - D^T(e^{\lambda\cdot})) - F^T(e^{\lambda\cdot}).$$

Lemma 2.1. *Let $\lambda \in \mathbb{R}$ and $\psi = (\psi_0, \psi_1) \in \mathcal{H}$. If $\zeta = (\zeta_0, \zeta_1) \in \mathscr{D}(\mathcal{A}^*)$ satisfies*

$$(\lambda I - \mathcal{A}^*)(\zeta_0, \zeta_1) = (\psi_0, \psi_1),$$

then

$$\Delta_T(\lambda)\zeta_0 = \psi_0 + \int_{-r}^0 e^{\lambda\theta}\psi_1(\theta)d\theta - D^T(e^{\lambda\cdot})\psi_0, \tag{2.10}$$

and

$$\begin{aligned} \zeta_1(\tau) = {} & \lambda D^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\zeta_0 + F^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\zeta_0 \\ & - D^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\psi_0 + \int_{-r}^\tau e^{\lambda(\theta-\tau)}\psi_1(\theta)d\theta, \qquad \tau \in [-r, 0], \end{aligned} \tag{2.11}$$

where $\mathbf{1}_E(\cdot)$ is the usual characteristic function, i.e., $\mathbf{1}_E(x)=1$ if $x\in E$ and $\mathbf{1}_E(x)=0$ if $x\notin E$.

Proof. By using the same notions and notations as in Proposition 2.1, we have for $\phi=(\phi_0,\phi_1)\in\mathscr{D}(\mathcal{A})$ and $(\varphi_0,\varphi_1)=(\lambda I-\mathcal{A})(\phi_0,\phi_1)\in\mathcal{H}$ that

$$\begin{aligned}
&\langle(\varphi_0,\varphi_1),((\lambda I-\mathcal{A})^{-1})^*(\psi_0,\psi_1)\rangle_{\mathcal{H}}\\
&\quad=\langle(\lambda I-\mathcal{A})^{-1}(\varphi_0,\varphi_1),(\psi_0,\psi_1)\rangle_{\mathcal{H}}\\
&\quad=\langle\phi_0,\psi_0\rangle+\int_{-r}^0\Big\langle e^{\lambda\theta}\phi_1(0)+\int_\theta^0 e^{\lambda(\theta-\tau)}\varphi_1(\tau)d\tau,\psi_1(\theta)\Big\rangle d\theta\\
&\quad=\Big\langle\phi_0,\psi_0+\int_{-r}^0 e^{\lambda\theta}\psi_1(\theta)d\theta\Big\rangle+\int_{-r}^0\langle D\phi_1,e^{\lambda\theta}\psi_1(\theta)d\theta\rangle\\
&\qquad+\int_{-r}^0\int_\theta^0\langle e^{\lambda(\theta-\tau)}\varphi_1(\tau),\psi_1(\tau)\rangle d\tau d\theta.
\end{aligned}\tag{2.12}$$

Let

$$\kappa=\psi_0+\int_{-r}^0 e^{\lambda\theta}\psi_1(\theta)d\theta,$$

then we have

$$\begin{aligned}
&\Big\langle\phi_0,\psi_0+\int_{-r}^0 e^{\lambda\theta}\psi_1(\theta)d\theta\Big\rangle+\int_{-r}^0\langle D\phi_1,e^{\lambda\theta}\psi_1(\theta)d\theta\rangle\\
&\quad=\langle\phi_1(0),\kappa\rangle-\langle D\phi_1,\kappa\rangle+\langle D\phi_1,\kappa-\psi_0\rangle\\
&\quad=\langle\phi_1(0),\kappa\rangle-\langle D\phi_1,\psi_0\rangle\\
&\quad=\langle\phi_1(0),\kappa\rangle-\Big\langle D(e^{\lambda\cdot})\phi_1(0)-D\Big(\int_\cdot^0 e^{\lambda(\cdot-\tau)}\varphi_1(\tau)d\tau\Big),\psi_0\Big\rangle,
\end{aligned}$$

which, in addition to Proposition 2.1, further implies that

$$\begin{aligned}
&\Big\langle\phi_0,\psi_0+\int_{-r}^0 e^{\lambda\theta}\psi_1(\theta)d\theta\Big\rangle+\int_{-r}^0\langle D\phi_1,e^{\lambda\theta}\psi_1(\theta)d\theta\rangle\\
&\quad=\Big\langle\lambda\Delta(\lambda)^{-1}\int_{-r}^0 D(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\varphi_1(\tau)d\tau\\
&\qquad+\Delta(\lambda)^{-1}\int_{-r}^0 F(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\varphi_1(\tau)d\tau+\Delta(\lambda)^{-1}\varphi_0,\kappa-D^T(e^{\lambda\cdot})\psi_0\Big\rangle\\
&\qquad-\int_{-r}^0\Big\langle\varphi_1(\tau),D^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\psi_0\Big\rangle d\tau\\
&\quad=\langle\varphi_0,(\Delta(\lambda)^{-1})^T(\kappa-D^T(e^{\lambda\cdot})\psi_0)\rangle\\
&\qquad+\int_{-r}^0\langle\varphi_1(\tau),\lambda D^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})(\Delta(\lambda)^{-1})^T(\kappa-D^T(e^{\lambda\cdot})\psi_0)\rangle d\tau\\
&\qquad+\int_{-r}^0\Big\langle\varphi_1(\tau),F^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})(\Delta(\lambda)^{-1})^T(\kappa-D^T(e^{\lambda\cdot})\psi_0)\Big\rangle d\tau\\
&\qquad-\int_{-r}^0\langle\varphi_1(\tau),D^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\psi_0\rangle d\tau.
\end{aligned}$$

Let $\zeta_0 = (\Delta(\lambda)^{-1})^T(\kappa - D^T(e^{\lambda\cdot})\psi_0)$, then we have

$$\begin{aligned}
&\left\langle \varphi_0, \psi_0 + \int_{-r}^{0} e^{\lambda\theta}\psi_1(\theta)d\theta \right\rangle + \int_{-r}^{0} \langle D\phi_1, e^{\lambda\theta}\psi_1(\theta)\rangle d\theta \\
&\quad = \langle \varphi_0, \zeta_0\rangle + \int_{-r}^{0} \langle \varphi_1(\tau), \lambda D^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\zeta_0\rangle d\tau \\
&\qquad + \int_{-r}^{0} \left\langle \varphi_1(\tau), F^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\zeta_0 \right\rangle d\tau - \int_{-r}^{0} \langle \varphi_1(\tau), D^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\psi_0\rangle d\tau.
\end{aligned} \tag{2.13}$$

Therefore, we have from Eqs. (2.12) and (2.13) that

$$\begin{aligned}
\zeta_1(\tau) &= \lambda D^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\zeta_0 + F^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\zeta_0 \\
&\quad - D^T(\mathbf{1}_{[\cdot,0]}e^{\lambda(\cdot-\tau)})\psi_0 + \int_{-r}^{\tau} e^{\lambda(\theta-\tau)}\psi_1(\theta)d\theta, \qquad \tau \in [-r, 0].
\end{aligned} \tag{2.14}$$

The proof is complete now. □

Now consider a particular form of the delay operators D and F. Precisely, for any $\varphi \in C([-r, 0], \mathbb{R}^n)$, assume that

$$D\varphi = D_1\varphi(-r) + \int_{-r}^{0} D_2(\theta)\varphi(\theta)d\theta, \qquad F\varphi = F_1\varphi(-r) + \int_{-r}^{0} F_2(\theta)\varphi(\theta)d\theta,$$

where $D_1, F_1 \in \mathcal{M}(\mathbb{R}^n)$ and $D_2, F_2 \in L^2([-r, 0], \mathcal{M}(\mathbb{R}^n))$. In this case, the following result which characterizes the infinitesimal generator $\mathcal{A}^*$ of the semigroup $e^{t\mathcal{A}^*}$ is a direct consequence of Lemma 2.1.

Theorem 2.1. *The infinitesimal generator $\mathcal{A}^*$ of the C_0-semigroup $e^{t\mathcal{A}^*}$ is given by*

$$\mathscr{D}(\mathcal{A}^*) = \left\{\zeta = (\zeta_0, \zeta_1) : \zeta_0 \in \mathbb{R}^n, \zeta_1 \in W^{1,2}([-r, 0], \mathbb{R}^n),\ \zeta_1(-r) = F_1^T\zeta_0 + D_1^T\zeta_1(0)\right\}, \tag{2.15}$$

and

$$\mathcal{A}^*\zeta = \left(\zeta_1(0), F_2^T(\cdot)\zeta_0 + D_2^T(\cdot)\zeta_1(0) - \zeta_1'(\cdot)\right) \quad \text{for} \quad \zeta = (\zeta_0, \zeta_1) \in \mathscr{D}(\mathcal{A}^*). \tag{2.16}$$

Proof. By using the same notions and notations as in Lemma 2.1, we have for $\lambda \in \mathbb{R}$ that

$$\Delta_T(\lambda)\zeta_0 = \kappa - D^T(e^{\lambda\cdot})\psi_0. \tag{2.17}$$

By taking $\lambda = 0$ in the definition of $\Delta_T(\lambda)$, Eqs. (2.17) and (2.11), we get

$$\begin{aligned}
\Delta_T(0)\zeta_0 &= -F_1^T\zeta_0 - \int_{-r}^{0} F_2^T(\theta)\zeta_0 d\theta \\
&= \psi_0 + \int_{-r}^{0} \psi_1(\tau)d\tau - D_1^T\psi_0 - \int_{-r}^{0} D_2^T(\theta)\psi_0 d\theta,
\end{aligned} \tag{2.18}$$

and for any $\tau \in [-r, 0]$,

$$\zeta_1(\tau) = \left(F_1^T + \int_{-r}^{\tau} F_2^T(\theta)d\theta\right)\zeta_0 - D_1^T\psi_0 - \int_{-r}^{\tau} D_2^T(\theta)d\theta\psi_0 + \int_{-r}^{\tau} \psi_1(\theta)d\theta. \tag{2.19}$$

From Eq. (2.18), we have

$$\psi_0 + \int_{-r}^{0} \psi_1(\theta)d\theta - D_1^T\psi_0 - \int_{-r}^{0} D_2^T(\theta)\psi_0 d\theta = -F_1^T\zeta_0 - \int_{-r}^{0} F_2^T(\theta)d\theta\zeta_0, \tag{2.20}$$

and from Eq. (2.19), we have, by letting $\tau = 0$, that

$$\zeta_1(0) = F_1^T\zeta_0 + \int_{-r}^{0} F_2^T(\theta)d\theta\zeta_0 - D_1^T\psi_0 - \int_{-r}^{0} D_2^T(\theta)d\theta\psi_0 + \int_{-r}^{0} \psi_1(\theta)d\theta. \tag{2.21}$$

Hence, we have from Eqs. (2.20) and (2.21) that

$$-\psi_0 = \zeta_1(0). \tag{2.22}$$

Let $\tau = -r$ in Eq. (2.19), then we get, in addition to Eq. (2.22), that

$$\zeta_1(-r) = F_1^T\zeta_0 - D_1^T\psi_0 = F_1^T\zeta_0 + D_1^T\zeta_1(0),$$

and by differentiating both sides of Eq. (2.19) with respect to τ and using Eq. (2.22), we get

$$-\psi_1(\tau) = F_2^T(\tau)\zeta_0 + D_2^T(\tau)\zeta_1(0) - \zeta_1'(\tau), \qquad \tau \in [-r, 0].$$

Now the proof is complete. □

3. Systems with delays in control

In this section, we shall apply the established theory in the previous sections to the so-called optimal control problem of a class of stochastic functional differential equations of neutral type in which delays are explicitly present in the control term. Precisely, let $G_1 \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^d)$, $G_2 \in L^2([-r, 0], \mathcal{M}(\mathbb{R}^n, \mathbb{R}^d))$. Consider the following stochastic control system in $\mathbb{R}^n$,

$$\begin{cases} x(t) = \phi_0 + D_1 x(t-r) + \int_0^t Gu_s ds + Bw(t) \text{for any}\, t \in [0, T], \\ x_0 = \phi_1, \phi = (\phi_0, \phi_1) \in \mathcal{H} = \mathbb{R}^n \times L^2([-r, 0], \mathbb{R}^n), \quad u_0 = \varphi \in L^2(\Omega \times [-r, 0], \mathbb{R}^d), \end{cases} \tag{3.1}$$

where $D_1 \in \mathcal{M}(\mathbb{R}^n)$, $B \in \mathcal{M}(\mathbb{R}^n, \mathbb{R}^m)$,

$$G\xi = G_1\xi(-r) + \int_{-r}^{0} G_2(\theta)\xi(\theta)d\theta \qquad \text{for any} \qquad \xi \in C([-r, 0], \mathbb{R}^d),$$

and $w(\cdot) = (w_1(\cdot), \dots, w_m(\cdot))^T$ is a standard m-dimensional Brownian motion. In association with Eq. (3.1), we can consider another stochastic system in $\mathbb{R}^n$ of the following form,

$$\begin{cases} z(t) = \phi_0 + D_1^T z(t-r) + \int_0^t Gu_s ds + \int_0^t Bdw(s), \qquad t \in [0, T], \\ z_0 = \phi_1, \phi = (\phi_0, \phi_1) \in \mathcal{H}, \quad u_0 = \varphi \in L^2(\Omega \times [-r, 0], \mathbb{R}^d). \end{cases} \tag{3.2}$$

The point is that we may associate Eq. (3.2) with an abstract stochastic controlled differential equation without delay on $\mathcal{H}$,

$$\begin{cases} dZ(t) = (\mathcal{A}^* Z(t) + \mathcal{G}u(t))dt + \mathcal{B}dw(t), \ t \in [0, T], \\ Z(0) = \phi = (\phi_0, \phi_1) \in \mathcal{H}, \end{cases} \tag{3.3}$$

where $\mathcal{A}^*$ is given as in Proposition 2.1, $\mathcal{B}: \mathbb{R}^m \to \mathcal{H}$ and $\mathcal{G}: \mathbb{R}^d \to \mathcal{H}$ are defined, respectively, by

$$\mathcal{B}: x \to (Bx, 0), \qquad \forall x \in \mathbb{R}^m,$$

and

$$\mathcal{G}: u \to (G_1 u, G_2(\theta)u), \ \theta \in [-r, 0], \ u \in \mathbb{R}^d.$$

Proposition 3.1. *Let $T \geq 0$ and $z(t, \phi)$, $t \geq 0$, be the unique solution of Eq. (3.2) with initial $\phi = (\phi_0, \phi_1) \in \mathcal{H}$ and adapted control process $u \in L^2(\Omega \times [-r, T], \mathbb{R}^d)$. Let*

$$Z_0(t) = z(t) - D_1^T z(t-r), \qquad t \in [0, T], \tag{3.4}$$

and

$$Z_1(t)(\theta) = \int_{-r}^{\theta} G_2(\tau)u(t+\tau-\theta)d\tau, \theta \in [-r, 0], t \in [0, T]. \tag{3.5}$$

Then for $t \geq r$, the process

$$Z(t) = (Z_0(t), Z_1(t)(\cdot)) = \left(z(t) - D_1^T z(t-r), \int_{-r}^{\cdot} G_2(\tau)u(t+\tau-\cdot)d\tau\right)$$

is the solution of the system (3.3).

Proof. Eq. (3.3) can be rewritten as

$$\begin{cases} dZ_0(t) = Z_1(t)(0)dt + G_1 u(t)dt + Bdw(t), \qquad t \in [0, T], \\ dZ_1(t)(\theta) = \Big(- dZ_1(t)(\theta)/d\theta + G_2(\theta)u(t)\Big)dt, \quad t \in [0, T], \quad \theta \in [-r, 0], \\ Z_0(0) = \phi_0, \quad Z_1(0)(\theta) = \phi_1(\theta), \quad \theta \in [-r, 0]. \end{cases} \tag{3.6}$$

The solution $Z(t)$ of Eq. (3.6) thus satisfies the following equations

$$Z_0(t) = \phi_0 + \int_0^t Z_1(s)(0)ds + \int_0^t G_1 u(s)ds + \int_0^t Bdw(s), \tag{3.7}$$

and for almost all $\theta \in [-r, 0]$,

$$\begin{aligned} Z_1(t)(\theta) &= [S(t)\phi_1](\theta) + \int_0^t [S(t-s)G_2(\cdot)u(s)](\theta)ds \\ &= [S(t)\phi_1](\theta) + \int_{\theta-t}^{\theta} G_2(\tau)u(t+\tau-\theta)d\tau, \end{aligned} \tag{3.8}$$

where $S(t)$ is the semigroup of truncated right shifts defined as

$$[S(t)\xi(\cdot)](\theta) = \begin{cases} \xi(\theta-t), & -r \leq \theta - t \leq 0, \\ 0, & \text{otherwise}, \end{cases}$$

for all $\xi \in L^2([-r, 0], \mathbb{R}^n)$. Hence, for $t \geq r$, the equality (3.8) can be rewritten, by the definition of $S(t)$, as

$$Z_1(t)(\theta) = \int_{-r}^{\theta} G_2(\tau)u(t+\tau-\theta)d\tau, \theta \in [-r, 0] \ a.e. \tag{3.9}$$

as claimed.

Next we intend to prove that $Z_0(t)$ satisfies the same type of integral equation as the mild solution $z(t)$ to the system (3.2) does. Indeed, letting $\theta = 0$ in Eq. (3.9) yields that for $t \in [r, T]$,

$$Z_1(t)(0) = \int_{-r}^{0} G_2(\tau)u(t+\tau)d\tau, \tag{3.10}$$

and substituting Eq. (3.10) into Eq. (3.7) yields that

$$z(t) - D_1^T z(t-r) = \phi_0 + \int_0^t G_1 u(s)ds + \int_0^t \int_{-r}^{0} G_2(\tau)u(s+\tau)d\tau ds + \int_0^t Bdw(s), \quad t \geq r,$$

which is Eq. (3.2). The proof is thus complete. □

Now let us consider the optimal control problem of Eq. (3.3). Let $T \geq r$ and for any initial data $\phi \in \mathcal{H}$ and adapted control process $u(\cdot)$ in $L^2(\Omega \times [-r, T], \mathbb{R}^d)$, let

$$Z(t, \phi, u) = \left(z(t) - D_1^T z(t-r), \int_{-r}^{\cdot} G_2(\tau)u(t+\tau-\cdot)d\tau \right), \qquad t \in [r, T],$$

denote the solution of Eq. (3.3) with initial $\phi = (\phi_0, \phi_1) \in \mathcal{H}$. The objective functional is given by

$$J(t, \phi, u) = \mathbb{E}^{\mathbb{P}}\left[g(Z(T, \phi, u)) + \int_t^T h(u(s))ds \right], \qquad t \in [0, T], \tag{3.11}$$

where $h: \mathbb{R}^d \to \mathbb{R}$ and $g: \mathcal{H} \to \mathbb{R}$ are appropriately given functions. The problem is to minimize the objective function $J(t, \phi, u)$ over all adapted processes $u \in L^2(\Omega \times [-r, T], \mathbb{R}^d)$. We also define the value function v^* for this problem as

$$v^*(t, \phi) = \inf_{u \in L^2(\Omega \times [-r,T], \mathbb{R}^d)} J(t, \phi, u), \qquad t \in [0, T]. \tag{3.12}$$

Moreover, we say that $u^* \in L^2(\Omega \times [-r, T], \mathbb{R}^d)$ is an *optimal control* if

$$v^*(t, \phi) = J(t, \phi, u^*), \qquad t \in [0, T], \qquad \phi \in \mathcal{H}.$$

In general, it is not easy to have a general existence theory of the optimal control problem (3.1) since this depends very much on the nature of value function (3.12) and operator $\mathcal{A}^*$ although it is possible for some particular systems such as Example 4.1 to have an explicit existence result. Next, by following the dynamical programming approach, we would like to characterize the value function v^* as the unique solution in a mild sense of the following HJB equation

$$\begin{cases} v_t'(t, \phi) + \frac{1}{2} Tr(Qv_{\phi\phi}''(t, \phi) + \langle \mathcal{A}^*\phi, v_\phi'(t, \phi)\rangle_{\mathcal{H}} \\ +H_0(v_\phi'(t, \phi)) = 0, \quad t \in [0, T], \quad \phi \in \mathscr{D}(\mathcal{A}^*), \\ v(T, \phi) = g(\phi), \quad \phi \in \mathcal{H}, \end{cases} \tag{3.13}$$

where v_ϕ', $v_{\phi\phi}''$ are the Fréchet derivatives of v, $Q = \mathcal{B}^*\mathcal{B}$ and

$$H_0(\psi) = \sup_{u \in \mathbb{R}^d} (\langle \mathcal{G}u, \psi \rangle_{\mathcal{H}} + h(u)), \qquad \psi \in \mathcal{H}.$$

Moreover, we would like to find a sufficient condition for optimality given in terms of v and an optimal control u^*.

Definition 3.1. A function v is said to be

(i) a *classical solution* of the HJB Eq. (3.13) if $v \in C^{1,2}([0,T]\times\mathcal{H})$ and v satisfies Eq. (3.13) pointwise;
(ii) an *integral solution* if $v \in C^{0,2}([0,T]\times\mathcal{H})$ and for $t\in[0,T]$ and $\phi \in \mathscr{D}(\mathcal{A}^*)$, one has

$$g(\phi) - v(t,\phi) + \int_t^T \left[\frac{1}{2}Tr(Qv''_{\phi\phi}(s,\phi)) + \langle \mathcal{A}^*\phi, v'_\phi(s,\phi)\rangle_{\mathcal{H}} + H_0(v'_\phi(s,\phi))\right]ds = 0.$$

Theorem 3.1 (Verification theorem)**.** *Let v be an integral solution of the HJB equation (3.13) and v^* be the value function of the optimal control problem in Eq. (3.12). Then*

(i) *$v \geq v^*$ on $[0,T]\times\mathcal{H}$;*
(ii) *If an adapted control process $u^* \in L^2(\Omega\times[-r,T],\mathbb{R}^m)$ is such that, at starting point (t,ϕ),*

$$\begin{aligned} H_0(v'_\phi(s,Z(s))) &= \sup_{u\in U}\{\langle \mathcal{G}u, v'_\phi(s,Z(s))\rangle_{\mathcal{H}} + h(u)\} \\ &= \langle \mathcal{G}u^*(s), v'_\phi(s,Z(s))\rangle_{\mathcal{H}} + h(u^*(s))\} \end{aligned}$$

for almost every $s\in[t,T]$, then this control is optimal and $v(t,\phi) = v^*(t,\phi)$.
(iii) *If we know a priori that $v^* = v$, then (ii) is a necessary and sufficient condition of optimality.*

Proof. We follow the same ideas as in [4] to sketch the proof. First, suppose that $\mathcal{A}^*$ is a bounded operator. We want to show that for every $(t,\phi)\in[0,T]\times\mathcal{H}$ and any $u \in L^2(\Omega\times[-r,T],\mathbb{R}^d)$, the following identity holds:

$$v(t,\phi) = J(t,\phi,u) + \mathbb{E}^{\mathbb{P}}\int_t^T \left[H_0(v'_\phi(s,Z(s))) - \langle \mathcal{G}u(s), v'_\phi(s,Z(s))\rangle_{\mathcal{H}} - h(u(s))\right]ds, \quad (3.14)$$

where $Z(s) = Z(s,t,\phi,u)$, $s\geq t$. Once this is proved, those three claims of Theorem 3.1 follow as instant consequences of the definition of the Hamiltonian H_0 of value function and of optimal strategy.

To this end, we first approximate v by a sequence of smooth functions v_n such that $v_n \to v$ and $(v_n)'_\phi \to v'_\phi$ as $n\to\infty$, which is possible, e.g., by using the same ideas of Goldys and Gozzi [3]. Then for $s\in[t,T]$, one may show that

$$dv_n(s,Z(s))/ds = (v_n)'_t(s,Z(s)) + \langle \mathcal{A}^*Z(s) + \mathcal{G}u(s), (v_n)'_\phi(s,Z(s))\rangle_{\mathcal{H}}. \quad (3.15)$$

Since v_n is a classical solution of a suitable approximation HJB equation (see again [3]), we can show that

$$(v_n)'_t(s,Z(s)) + \langle \mathcal{A}^*Z(s), (v_n)'_t(s,Z(s))\rangle_{\mathcal{H}} = -H_0((v_n)'_\phi(s,Z(s))) - f_n(s,Z(s)),$$

where f_n is a term appearing in the approximating HJB such that $f_n\to 0$ as $n\to\infty$ (see again [3]). Substituting in Eq. (3.15) and then adding and subtracting $h(u(s))$, we obtain

$$\begin{aligned} dv_n(s,Z(s))/ds &= \langle u(s), \mathcal{G}^*(v_n)'_\phi(s,Z(s))\rangle_{\mathcal{H}} - H_0((v_n)'_\phi(s,Z(s))) - f_n(s,Z(s)) \\ &= \langle u(s), \mathcal{G}^*(v_n)'_\phi(s,Z(s))\rangle_{\mathcal{H}} + h(u(s)) - H_0((v_n)'_\phi(s,Z(s))) \\ &\quad - f_n(s,Z(s)) - h(u(s)). \end{aligned}$$

Integrating from t to T, we get

$$\begin{aligned} &v_n(T, Z(T)) - v_n(t, \phi) + \int_t^T [f_n(s, Z(s)) + h(u(s))]ds \\ &= \int_t^T [\langle u(s), \mathcal{G}^*((v_n)'_\phi(s, Z(s)))\rangle_{\mathcal{H}} + h(u(s)) - H_0((v_n)'_\phi(s, Z(s)))]ds, \end{aligned}$$

which immediately yields that

$$\begin{aligned} v_n(t, \phi) = &\mathbb{E}^{\mathbb{P}}\bigg[v_n(T, Z(T)) + \int_t^T h(u(s))ds\bigg] + \mathbb{E}^{\mathbb{P}}\int_t^T f_n(s, Z(s))ds \\ &+\mathbb{E}^{\mathbb{P}}\int_t^T \bigg[H_0((v_n)'_\phi(s, Z(s))) - \langle u(s), \mathcal{G}^*(v_n)'_\phi(s, Z(s))\rangle_{\mathcal{H}} - h(u(s))\bigg]ds. \end{aligned}$$

This gives the desired equality (3.14) after passing to the limit as $n \to \infty$. Last, the case of unbounded $\mathcal{A}^*$ can be dealt with by approximating $\mathcal{A}^*$ with its Yosida approximations $\mathcal{A}^*_n$, and then passing to the limit as $n \to \infty$. The proof is thus complete. □

4. An example with explicit solution

In Eq. (3.2), if D_1 is a symmetric matrix, then Eq. (3.2) is as same as Eq. (3.1). In this case, to study the optimal control problem of Eq. (3.1), it suffices to consider by virtue of Proposition 3.1 the corresponding problem of system (3.3). In the remainder, we discuss an example to obtain its explicit optimal control to illustrate the theory in this work.

Example 4.1. Consider the following one dimensional SDDE of neutral type

$$\begin{cases} z(t) = \phi_0 + z(t-r) + \displaystyle\int_0^t b_0 u(s-r)ds + \int_0^t \int_{-r}^0 b_1(\theta)u(s+\theta)d\theta ds + w(t), \quad t \geq 0, \\ z_0 = \phi_1, \phi = (\phi_0, \phi_1) \in \mathbb{R} \times L^2([-r, 0], \mathbb{R}), u_0 = \varphi \in L^2([-r, 0], \mathbb{R}), \end{cases} \tag{4.1}$$

where $b_0 \in \mathbb{R}$, $b_1 \in L^2([-r, 0], \mathbb{R})$ and w is a standard real Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$. Let $T \geq r$ and we consider the objective functional

$$J(u(\cdot)) = \mathbb{E}^{\mathbb{P}}\bigg[g_0(z(T) - z(T-r)) + \int_0^T h_0(u(t))dt\bigg], \tag{4.2}$$

where $g_0(\cdot)$ and $h_0(\cdot)$ are two appropriately given functions, and the dynamics of z is determined by Eq. (4.1). The problem we are concerned about is the minimization of the objective functional $J(\cdot)$ over all the so-called admissible controls u.

Let $\mathcal{H} = \mathbb{R} \times L^2([-r, 0], \mathbb{R})$. We can formulate this example into the abstract form Eqs. (3.3) and (3.11) where the functions $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathcal{H} \to \mathbb{R}$ are given by

$$\begin{aligned} &h(u) = h_0(u), \qquad u \in \mathbb{R}, \\ &g(\phi) = g(\phi_0, \phi_1) = g_0(\phi_0), \qquad \phi \in \mathcal{H}. \end{aligned}$$

To obtain an explicit solution, let us consider on this occasion that

$$h_0(u) = -\beta u^2, \quad \beta > 0, \qquad \forall u \in \mathbb{R},$$

and

$$g_0(\phi_0) = \gamma\phi_0, \quad \gamma > 0, \qquad \forall\, \phi_0 \in \mathbb{R}.$$

We guess a solution of the HJB equation (3.13) of the form

$$v(t,\phi) = \langle \xi(t), \phi\rangle_{\mathcal{H}} + c(t), \qquad t \in [0,T], \quad \phi \in \mathcal{H},$$

where $\xi(\cdot) : [0,T] \to \mathcal{H}$ and $c(\cdot) : [0,T] \to \mathbb{R}$ are two functions to be determined later on. Then we have for $\psi \in \mathcal{H}$,

$$H_0(\psi) = \sup_{u\in\mathbb{R}} \big(\langle \mathcal{G}u, \psi\rangle_{\mathcal{H}} + h_0(u)\big) = \sup_{u\in\mathbb{R}} \big(\langle \mathcal{G}, \psi\rangle_{\mathcal{H}} u - \beta u^2\big) = \frac{(\langle \mathcal{G}, \psi\rangle_{\mathcal{H}}^+)^2}{4\beta},$$

where the supremum is reached at $u = u^* = \langle \mathcal{G}, \psi\rangle_{\mathcal{H}}^+/(2\beta)$. Here $a^+ = a$ for $a \geq 0$ and $a^+ = 0$ for $a < 0$. For $t \in [0, T]$ and $\phi \in \mathcal{H}$ we have

$$\begin{cases} v'_t(t,\phi) = \langle \xi'(t), \phi\rangle_{\mathcal{H}} + c'(t), \\ v'_\phi(t,\phi) = \xi(t), \\ v''_{\phi\phi} = 0, \end{cases} \tag{4.3}$$

and Eq. (3.13) turns out to be

$$\begin{cases} \langle \xi'(t), \phi\rangle_{\mathcal{H}} + c'(t) + \langle \xi, \mathcal{A}^*\phi\rangle_{\mathcal{H}} + \dfrac{(\langle \mathcal{G}, \xi(t)\rangle_{\mathcal{H}}^+)^2}{4\beta} = 0, \quad t \in [0,T], \phi \in \mathscr{D}(\mathcal{A}^*), \\ \langle \xi(T), \phi\rangle_{\mathcal{H}} + c(T) = \gamma\phi_0, \qquad \phi \in \mathcal{H}. \end{cases} \tag{4.4}$$

Instead of dealing with Eq. (4.4), by assuming $\xi(t) \in \mathscr{D}(\mathcal{A})$, the domain of $\mathcal{A}$, for almost all $t \in [0, T]$, we can equivalently consider the equations

$$\begin{cases} \langle \xi'(t), \phi\rangle_{\mathcal{H}} + c'(t) + \langle \mathcal{A}\xi, \phi\rangle_{\mathcal{H}} + \dfrac{(\langle \mathcal{G}, \xi(t)\rangle_{\mathcal{H}}^+)^2}{4\beta} = 0, \quad t \in [0,T], \phi \in \mathcal{H}, \\ \langle \xi(T), \phi\rangle_{\mathcal{H}} + c(T) = \gamma\phi_0, \qquad \phi \in \mathcal{H}. \end{cases} \tag{4.5}$$

Since Eq. (4.5) must hold for all $\phi \in \mathcal{H}$, then it implies that

$$\begin{cases} \xi'(t) + \mathcal{A}\xi(t) = 0, \quad t \in [0,T], \\ \xi(T) = (\gamma, 0), \end{cases} \tag{4.6}$$

and

$$\begin{cases} c'(t) + \dfrac{(\langle \mathcal{G}, \xi(t)\rangle_{\mathcal{H}}^+)^2}{4\beta} = 0, \quad t \in [0,T], \\ c(T) = 0, \end{cases} \tag{4.7}$$

in which the first equation in Eq. (4.6) yields the equation

$$\begin{cases} \xi'_0(t) = 0, \ t \in [0,T], \\ \xi_0(T) = \gamma, \end{cases}$$

whose solution is given by $\xi_0(t) = \gamma$ for all $t \in [0, T]$. From the other equations in Eq. (4.6), we have

$$\begin{cases} \frac{\partial \xi_1}{\partial t}(t,\theta) + \frac{\partial \xi_1}{\partial \theta}(t,\theta) = 0,\ \theta \in [-r,0], \qquad t \in [0,T], \\ \xi_1(T,\theta) = 0,\ \theta \in [-r,0], \\ \xi_0(t) = \xi_1(t,0) - \xi_1(t,-r), \qquad t \in [0,T], \end{cases} \tag{4.8}$$

which immediately implies that

$$\xi_1(t,\theta) = \begin{cases} \xi_1(t-\theta,0), & \text{if}\, t-\theta \in [0,T], \\ 0, & \text{if}\, t-\theta \notin [0,T], \end{cases}$$

and

$$\begin{aligned} \xi_1(t,0) &= \xi_1(t+r,0) + \xi_0(t) \\ &= \xi_1(t+r,0) + \gamma \\ &= \xi_1(t+2r,0) + \gamma + \gamma \\ &= \dots . \end{aligned}$$

Hence, by solving these equations, we have that

$$\begin{aligned} &\xi_1(t,0) = 0, \qquad \forall t \notin [0,T], \\ &\xi_1(t,0) = \gamma, \qquad \forall t \in [T-r,T], \\ &\xi_1(t,0) = \gamma + \gamma, \qquad \forall t \in [T-2r,T-r], \\ &\qquad \dots\dots \\ &\xi_1(t,0) = \sum_{k=0}^{[T/r]} \gamma, \qquad \forall t \in [0, T-[T/r]r], \end{aligned}$$

where $[T/r]$ is the biggest integer less than or equal to T/r, and

$$\xi_0(t) = \gamma, \qquad t \in [0,T].$$

Hence, the optimal solution is

$$\xi(t) = (\xi_0(t), \xi_1(t,\theta)) = (\gamma, \xi_1(t-\theta,0)), \qquad \theta \in [-r,0], \qquad t \in [0,T], \tag{4.9}$$

where

$$\begin{aligned} \xi_1(t-\theta,0) &= 0, \qquad \forall t \notin [0,T], \\ \xi_1(t-\theta,0) &= \gamma, \qquad \forall t-\theta \in [T-r,T], \quad \theta \in [-r,0], \\ &\dots\dots \\ \xi_1(t-\theta,0) &= \sum_{k=0}^{[T/r]} \gamma \quad \text{for} \quad t-\theta \in [0, T-[T/r]r], \quad \theta \in [-r,0]. \end{aligned}$$

Since a minimizer of the current-value Hamiltonian is given by $u^*(t) = \langle \mathcal{G}, v'_\phi(t)\rangle^+_{\mathcal{H}}/(2\beta)$ where $\mathcal{G} = (b_0, b_1(\cdot))$, then it is immediately to see from Eq. (4.3) that the control

$$u^*(t) = \frac{\langle \mathcal{G}, v'_\phi(t)\rangle^+_{\mathcal{H}}}{2\beta} = \frac{\langle \mathcal{G}, \xi(t)\rangle^+_{\mathcal{H}}}{2\beta}, \qquad t \in [0,T]. \tag{4.10}$$

Hence, substituting Eq. (4.9) into Eq. (4.10) we obtain the explicit form of the optimal control u^*. For example, if we consider the particular case $b_1(\cdot) \equiv 0$, $b_0 > 0$, the optimal control is

$$u^*(t) = \frac{b_0\gamma}{2\beta}, \qquad t \in [0,T].$$

References

[1] R. Brayton, Small signal stability criterion for electrical networks containing lossless transmission lines, IBM J. Res. Dev. 12 (1968) 431–440.

[2] J.A. Burns, T.L. Herdman, H.W. Stech, Linear functional differential equations as semigroups in product spaces, SIAM J. Math. Anal. 14 (1983) 98–116.

[3] B. Goldys, F. Gozzi, Second order parabolic HJB equations in hilbert spaces and stochastic control: l^2_μ approach, Stoch. Proc. Appl. 116 (2006) 1932–1963.

[4] F. Gozzi, C. Marinelli, Stochastic optimal control of delay equations arising in advertising models, in: Stochastic Partial Differential Equations – VII, in: Lecture Notes Pure Applied Mathematics, vol. 245, Chapman & Hall/CRC, Boca Raton, 2006, pp. 133–148.

[5] F. Gozzi, C. Marinelli, S. Savin, On controlled linear diffusions with delay in a model of optimal advertising under uncertainty with memory effects, J. Optim. Theory Appl. 142 (2009) 291–321.

[6] J. Hale, S. Lunel, Introduction to Functional Differential Equations., Springer-Verlag, Berlin, New York, 1993.

[7] K. Ito, T.J. Tarn, A linear quadratic optimal control for neutral systems, Nonlinear Analysis: Theory, Methods and Applications 9 (1985) 699–727.

[8] F. Kappel, K.P. Zhang, Equivalence of functional-differential equations of neutral type and abstract cauchy problems, Monatshefte für Math. 101 (1986a) 115–133.

[9] F. Kappel, K.P. Zhang, On neutral functional differential equations with nonatomic difference operator, J. Math. Anal. Appl. 113 (1986b) 311–343.

[10] V.B. Kolmanovskii, A. Myshkis, Introduction to the Theory and Applications of Functional Differential Equations., Kluwer Academic Publishers, 1999.

[11] K. Liu, Stationary Solutions of Neutral Stochastic Partial Differential Equations with Delays in the Highest-Order Derivatives, To appear in: Discrete and Continuous Dynamical Systems Series B, 23, 2018.

[12] X.R. Mao, Stochastic Differential Equations and Applications, Second Edition, Woodhead Publishing, Oxford, 2007.

[13] D. Salamon, Control and Observation of Neutral Systems, Research Notes in Mathematics, vol. 91, Pitman Advanced Publishing Program, London, 1984.

[14] R. Vinter, R. Kwong, The infinite time quadratic control problem for linear system with state and control delays: an evolution equation approach, SIAM J. Control Optim. 19 (1981) 139–153.